J. Fluid Mech. (1964), *vol.* 20, *part* 1, *pp.* 121–128
Printed in Great Britain

The behaviour of clusters of spheres falling in a viscous fluid

Part 1. Experiment

By K. O. L. F. JAYAWEERA, B. J. MASON
Physics Department, Imperial College, London
AND G. W. SLACK
Chemical Defence Experimental Establishment, Porton, Nr Salisbury, Wiltshire

(Received 17 December 1963)

The sedimentation of small clusters of uniform spheres, falling freely through a viscous liquid, has been studied with Reynolds numbers (based on diameter of the sphere and its velocity of free fall in the unbounded fluid) of individual spheres ranging from 10^{-4} to 10. The fall velocity of a cluster is, in all cases, greater than that of individual spheres, the more so when the spheres are closer together. Two spheres falling side-by-side rotate inwards and separate as they fall if $Re > 0{\cdot}05$, but no rotation nor separation is observed for $Re < 0{\cdot}03$. When equal-sized spheres of $Re > 1$ fall vertically one behind the other, the rear sphere is accelerated into the wake of the leader, rotates round it and separates from it when the line of centres is horizontal. If two spheres of unequal size but the same individual terminal velocity fall together, the smaller always travels faster than the larger. When three similar equally spaced spheres are dropped in a horizontal line, they interchange positions but do not separate when $0{\cdot}06 < Re < 0{\cdot}16$. But, if $0{\cdot}16 < Re < 3$, one sphere is always left behind; which sphere depends critically upon the initial spacings. If three to six equal spheres, of $0{\cdot}06 < Re < 7$, start falling as a compact cluster, they eventually draw level and arrange themselves in the same horizontal plane at the vertices of a regular polygon. The polygon expands at a decreasing rate during fall. When three spheres are arranged initially in a horizontal isosceles triangle, the spheres oscillate about their equilibrium positions but eventually the spheres form a stable equilateral triangle. If $Re > 7$, or the cluster contains 7 or more equal spheres, it shows no tendency to form a regular polygon but breaks up into two or more groups. A regular heptagon, and a hexagon with an additional sphere at its centre, are also unstable.

1. Introduction

The fall of single spheres and the sedimentation of a uniform suspension of many particles in a viscous medium have been studied extensively but the intermediate case of a falling cluster containing only a few spheres has apparently received little attention.

The behaviour of a pair of spheres of very low Reynolds number falling in close proximity through a viscous fluid was first treated theoretically by Smoluchowski (1911, 1912) by a method which is equivalent to superimposing the motion produced by the two spheres in isolation. He calculated the first terms in an expansion in powers of a/s, the radius of the sphere divided by the separation between centres, and his result is valid only for small values of this ratio. By introducing terms to satisfy the boundary conditions on one sphere and then on the other, the drag can be found by iteration to any degree of accuracy. This was done by Faxén (1925) who found the drag for two spheres moving along their line of centres by including terms of up to $(a/s)^5$. Stimson & Jeffery (1926) also solved this problem for two spheres falling one behind the other where there is axial symmetry. Kynch (1959) extended the analysis to include higher powers of a/s and to facilitate closer comparison with experiments for small separations. He also treated the case of unequal spheres. Hocking (1958) has expressed the drag for two spheres falling in line, either one behind the other, or side-by-side, as a series in a/s to as high a degree as required, and has evaluated the drag as a function of separation by retaining terms of up to $(a/s)^7$.

The main predictions of these workers may be summarized as follows.

(i) Pairs of spheres always fall faster under gravity than do single spheres falling alone.

(ii) This enhancement in the rate of fall is more marked when the spheres are close together.

(iii) Pairs of equal spheres falling together maintain a constant separation and orientation.

(iv) The members of a pair fall vertically only if the line joining their centres is either vertical or horizontal. Otherwise their velocity has a component in a downward sense along the line joining their centres.

(v) If two spheres of unequal size but having the same individual terminal velocity fall together, the smaller will always move faster than the larger.

Eveson, Hall & Ward (1959) claimed to have verified the first four of these predictions and also to have shown that the spheres fall without rotation but, after the publication of some observations by Matthews & Smith (1960), Eveson (1960) reported having observed a slow rotation with a pair of falling spheres.

Since the behaviour of more than two spheres falling under the influence of both viscous and inertial forces appears to defy detailed mathematical analysis, it was decided to carry out some simple experiments with spheres falling through a viscous liquid. Accordingly, observations were made with spheres having individual Reynolds numbers (based on sphere diameter and velocity of free fall) ranging from 10^{-4} to 10. The lower values allow one to test the theories based on the total neglect of inertial forces while the higher values of Re extend into régimes where both viscous and inertial terms are significant.

2. Experimental method

Experiments were conducted at Porton in a Perspex tank, 20 cm × 20 cm × 90 cm deep, containing castor oil (density 0·97 g cm^{-3} at 20 °C). The tank was surrounded by a thermostatically controlled lagged cabinet provided with glass windows and this cabinet was itself contained in another thermostatically controlled chamber. In this way, the temperatûre of the liquid could be held constant to within $\pm 0{\cdot}01$ °C for several hours and no convection currents were ever detected. Spheres of steel (density $7{\cdot}8\,\mathrm{g\,cm^{-3}}$), aluminium ($2{\cdot}7\,\mathrm{g\,cm^{-3}}$), bakelite ($1{\cdot}3\,\mathrm{g\,cm^{-3}}$), ebonite ($1{\cdot}2\,\mathrm{g\,cm^{-3}}$) and Perspex ($1{\cdot}19\,\mathrm{g\,cm^{-3}}$) ranging in diameter from 0·03 to 12·5 cm were used. Suitable choices of sphere size and density, and of liquid density and viscosity, enabled observations to be made of spheres with Reynolds numbers ranging from about 10^{-4} to 10.

The Imperial College experiments were carried out in a glass tank 30 cm $\times$ 30 cm $\times$ 60 cm deep containing liquid paraffin (density $0{\cdot}88\,\mathrm{g\,cm^{-3}}$ at 20 °C). This tank was chosen after several tests with larger and smaller tanks showed that there were no undesirable wall effects with a tank of this convenient size. Spheres of polyethylene ($0{\cdot}98\,\mathrm{g\,cm^{-3}}$), polystyrene ($1{\cdot}05\,\mathrm{g\,cm^{-3}}$), nylon ($1{\cdot}14\,\mathrm{g\,cm^{-3}}$) and Perspex ($1{\cdot}19\,\mathrm{g\,cm^{-3}}$) ranging from $\frac{1}{8}$ in. to $\frac{3}{8}$ in. at intervals of $\frac{1}{16}$ in. were used. The Reynolds number for a constant geometry could be varied by heating or cooling the paraffin and thereby changing its kinematic viscosity which varied from 88 cS at 36 °C to 340 cS at 15 °C. With this arrangement it was possible to work with spheres of Reynolds number, defined as ud/ν, d being the diameter of the sphere, u its terminal velocity in an infinite medium and ν the kinematic viscosity of the fluid, ranging from $< 0{\cdot}01$ to 10.

Two methods of releasing a small cluster of spheres were used. The first, designed to produce a close but random arrangement of spheres, employed a $\frac{3}{4}$ in. thick aluminium disk which rotated in the horizontal plane over, and almost in contact with, a flat stationary plate over the centre of the tank. The disk contained a number of holes of various sizes and the required number of spheres was placed in whichever of the holes they most nearly filled to a height equal to its diameter. The disk was then rotated until the hole containing the spheres was brought close to the edge of a slot in the lower plate. The cluster was then released by rotating the disk rapidly through a small angle. The spheres fell about 2 cm before entering the liquid. This method greatly reduced the entrainment of air bubbles in the cluster; in any case, clusters containing bubbles were not used.

In the second method, the spheres were held by suction in holes drilled in a flat plate. The plate was suspended just below the surface of the liquid and the spheres fell away immediately the partial vacuum was released. This method enables coplanar clusters of known geometry and spacing to be produced.

3. Sedimentation of a pair of spheres

(*a*) *Equal-sized spheres falling side-by-side*

Over the whole range of Reynolds numbers, the rate of fall of both spheres is greater, in all cases, than that of either sphere falling individually, and this enhancement in speed of fall is greater when the spheres are close together.

For $Re < 0{\cdot}03$, the spheres show no tendency to separate or to rotate but for $Re > 0{\cdot}05$, each sphere rotates inwards (in the sense shown in figure 2, plate 1) about a horizontal axis through its centre and normal to the line of centres. The rate of rotation increases with increasing Reynolds number but decreases as the spheres separate. The rotation continues until the separation reaches a maximum limiting value, which is a decreasing function of the Reynolds number as shown in figure 1, and ceases when the separation exceeds this value. Figure 2, plate 1, shows the rotation and separation of a pair of half-painted Perspex spheres ($d = \frac{3}{8}$ in., $Re \simeq 0{\cdot}1$) falling through castor oil and photographed at intervals of 2 sec.

(*b*) *Equal-sized spheres falling vertically one behind the other*

When $Re > 1$, the rear sphere becomes accelerated in the wake of the front sphere and tends to overtake it. When Re exceeds 4 this acceleration is already noticeable when the spheres are ten diameters apart. At large distances apart the

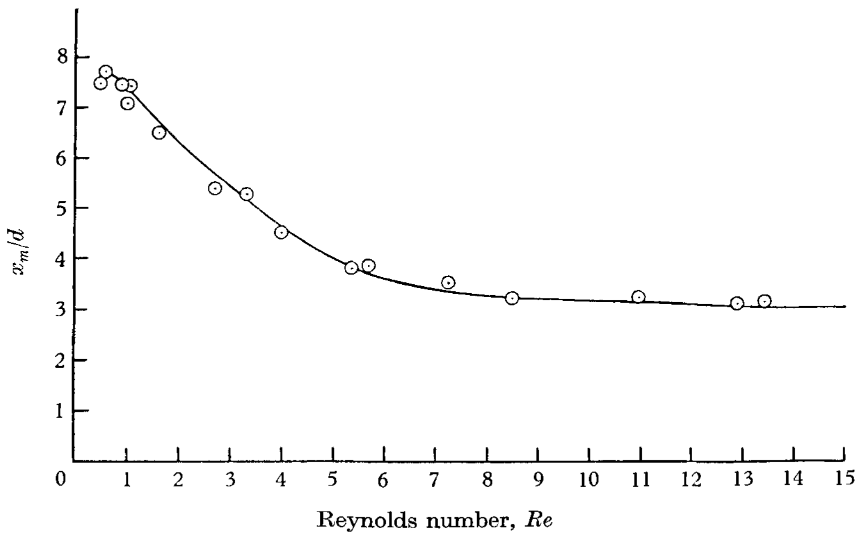

FIGURE 1. The maximum limiting separation x_m of two equal spheres falling side-by-side as a function of Reynolds number. d = diameter of sphere.

velocity of approach varies inversely as the separation as shown theoretically by Goldstein (1929) and Pearcey & McHugh (1955), but settles down to a nearly constant value at small separations—see figure 3. The relative velocity on apparent impact is about one-half the terminal velocity of an individual sphere. Until this stage neither sphere rotates but, now, the rear sphere slides round the leader and, when the line of centres becomes horizontal, the spheres separate, rotate in opposite directions, and continue to diverge laterally as they fall just as in (*a*).

(*c*) *Two equal spheres with line of centres inclined to horizontal*

A pair of equal spheres, each with $Re > 1$ and initially in different horizontal planes, appear to slide along the line of centres as well as falling vertically. They tend to occupy the same horizontal plane, and, having done so, remain like this but diverge laterally at a steadily decreasing rate.

(d) Pairs composed of unequal-sized spheres having the same individual terminal velocity

By adjusting the temperature of the castor oil an 8 mm ebonite sphere was made to fall at the same terminal velocity as a 1·58 mm steel sphere, and a 24·63 mm Perspex sphere to fall at the same velocity as a 3·97 mm steel sphere. In both cases, the smaller sphere of the pair always fell more rapidly than the larger, just as predicted by Kynch (1959), and this result was made unambiguous by adjusting the temperature of the liquid so that the larger sphere had a slightly *higher* terminal velocity.

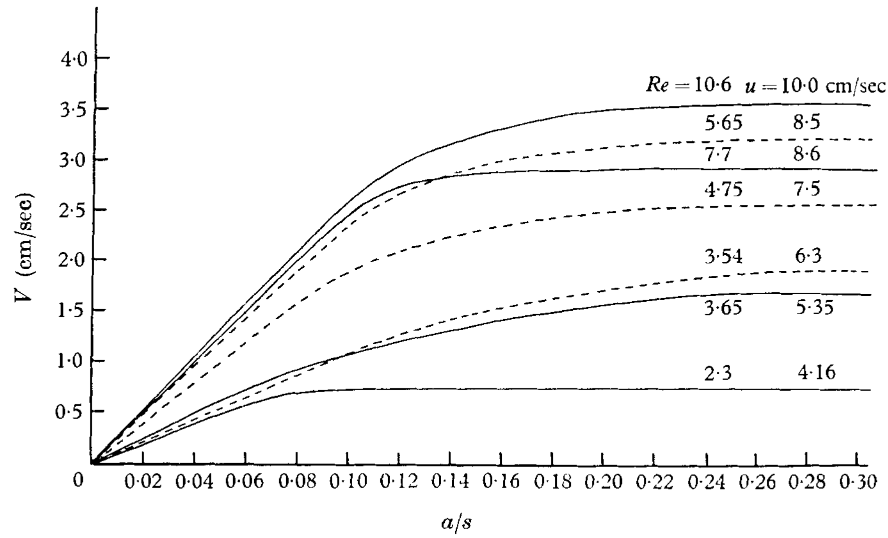

FIGURE 3. The relative velocity of approach, V, for two equal spheres falling one behind the other, plotted as a function of a/s for various values of the Reynolds number. u is the terminal velocity of an individual sphere, a its radius, and s the vertical separation of the two spheres. ——, at 88 cS; ----, at 140 cS.

The relative motion of the two spheres, with Re 0·3 and 1·8, is shown in figure 4. This shows the successive positions of the smaller sphere relative to the larger as obtained from a series of photographs taken at regular intervals. The left-hand picture shows the smaller sphere being accelerated into the wake of the larger, colliding with it, rolling round it, and then separating from it. In the right-hand picture, the smaller sphere is first attracted towards the larger but, on coming level with it, is repelled by it. Both spheres showed rotation which increased as the spheres approached each other and died away as they separated.

4. Equal-sized spheres released in a horizontal straight line

(a) $0{\cdot}06 < Re < 0{\cdot}16$

When three spheres, initially in contact or equally spaced within 6 diameters, are released in a horizontal straight line, the centre sphere moves slightly ahead, one of the laggards then moves between the other two, and the third (now trailing) passes between the other two. This interchanging of positions continues throughout the fall, but the spheres keep clòse together and do not separate. The sphere temporarily in the lead stops rotating. If the spheres are not equally spaced initially, one sphere is always left behind. If the spheres be numbered and separated as

$$(1) \leftarrow a \rightarrow (2) \leftarrow b \rightarrow (3),$$

which sphere is left behind depends critically upon the ratio b/a as follows:

b/a	$< 1{\cdot}17$	$1{\cdot}20$–$1{\cdot}33$	$1{\cdot}33$–$1{\cdot}40$	$1{\cdot}50$	$1{\cdot}60$–$2{\cdot}0$	$> 2{\cdot}0$
sphere left behind	1	2	1	3	2	3

For $b/a < 2$, one sphere is left behind after one or two interchanges of position, but if $b/a > 2$, sphere 3 is left behind from the start.

Two slightly separated pairs initially in a horizontal straight line diverge as they fall, each pair rotating as a doublet. Ultimately the members of a pair separate. With five or six spheres the outer ones move out, re-enter the cluster from the rear, and cause it to break up.

(*b*) $0{\cdot}16 < Re < 3$

If between three and six contacting spheres are released in a straight line they separate and eventually form a regular polygon, all the spheres lying in the same horizontal plane.

(*c*) $Re > 3$

The polygon is not regular and progressively breaks down as the Reynolds number is increased until, with $Re > 7$, the spheres appear to repel each other and separate but show no tendency to form a polygon.

5. Clusters of 3 to 6 equal spheres

$0{\cdot}06 < Re < 7$

When three to six equal spheres, starting as a compact cluster, fall together we observe that:

(i) Their speed of fall is greater than that of a single sphere.

(ii) This enhancement of the rate of fall is greater the more compact the cluster.

(iii) Even if the spheres are initially staggered by a few diameters they eventually draw level and ultimately arrange themselves in the same plane at the vertices of a regular polygon; see figure 5, plate 2.

(iv) The polygon expands slowly and at a decreasing rate during fall.

(v) When three spheres are arranged initially in a horizontal isosceles triangle the apex sphere oscillates in a decreasing vertical spiral relative to the other two, which execute linear oscillations along the horizontal line of centres and move apart as the apex sphere moves towards them. These oscillations gradually die out and the three spheres form a stable equilateral triangle. During the formation of 4-, 5-, or 6-sided polygons the individual spheres again oscillate about their equilibrium positions but eventually achieve a stable, regular configuration.

(vi) The final configuration is achieved more slowly with a larger number of more widely separated spheres.

(vii) During the early life of the polygon each sphere rotates inwards about a horizontal axis that is normal to a line joining the centre of the sphere to the centre of the polygon.

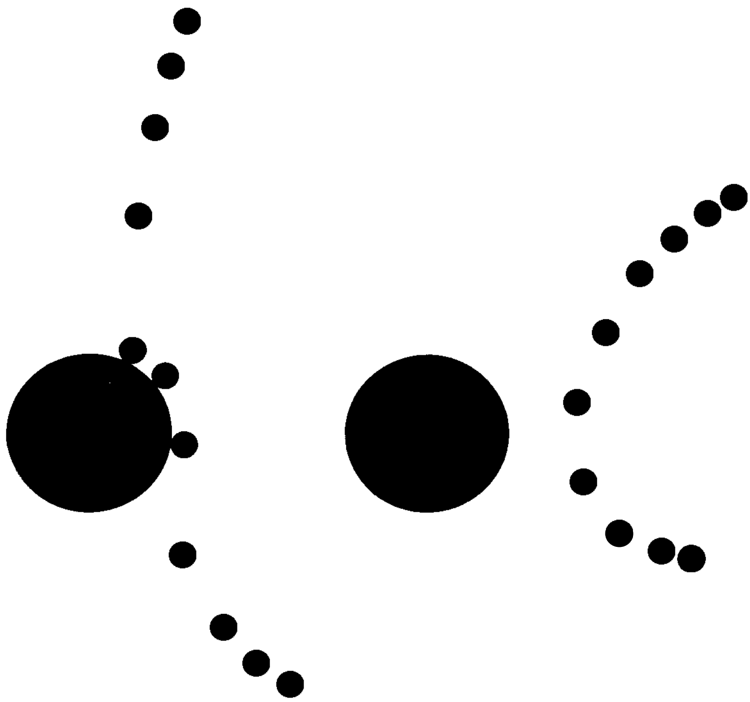

FIGURE 4. Relative motion of two unequal-sized spheres having the same terminal velocity. The diagram shows the position of the smaller sphere at equal intervals of time for two distinct cases.

(viii) When the separation of the spheres exceeds a certain value (about 6 diameters at $Re \simeq 1$ and 3 diameters at $Re \simeq 7$), rotation ceases but separation continues. Regular polygons are not formed if the *initial* separation of the spheres exceeds the critical distance.

(ix) If the spheres composing a cluster are arranged asymmetrically the densest portion of the cluster travels fastest. This causes a tilting of the cluster which then appears to slide as a whole along the line of tilt.

$$Re < 0{\cdot}06$$

The spheres tend to follow their initial configuration but fall faster than isolated spheres. They show no tendency to form a regular polygon and even if released in such a pattern they are susceptible to small perturbations.

$$Re > 7$$

The spheres of a cluster separate quickly with a sudden onset of rotation, but this soon ceases, and there is no tendency to form regular polygons.

6. Clusters containing more than six spheres

A compact cluster containing 7 or more equal spheres shows no tendency to form a regular polygon but tends to break up into two or more groups. A cluster arranged initially at the apices of a regular heptagon is unstable. For $Re < 0{\cdot}1$ the heptagon becomes distorted and for larger values of Re it breaks up during fall. If an additional sphere is placed at the centre of a regular hexagon it moves ahead and to one side. The hexagon then tilts towards this sphere which re-enters the hexagon from the rear and causes it to break up.

REFERENCES

Eveson, G. F. 1960 *Brit. J. Appl. Phys.* **11**, 88.
Eveson, G. F., Hall, E. W. & Ward, S. G. 1959 *Brit. J. Appl. Phys.* **10**, 43.
Faxén, H. 1925 *Arkiv. Math., Astron. Fysik*, **19***a*, 13.
Goldstein, S. 1929 *Proc. Roy. Soc.* A, **123**, 216.
Hocking, L. M. 1958 Ph.D. Thesis, London University.
Kynch, G. J. 1959 *J. Fluid Mech.* **5**, 193.
Matthews, H. W. & Smith, F. B. *Brit. J. Appl. Phys.* **11**, 87.
Pearcey, T. & McHugh, B. 1955 *Phil. Mag.* **46**, 783.
Smoluchowski, M. S. 1911 *Bull. Acad. Sci. Cracow*, **1** a, 28.
Smoluchowski, M. S. 1912 *Proc. 5th Int. Congr. Math.* **2**, 192.
Stimson, M. & Jeffery, G. B. 1926 *Proc. Roy. Soc.* A, **111**, 110.

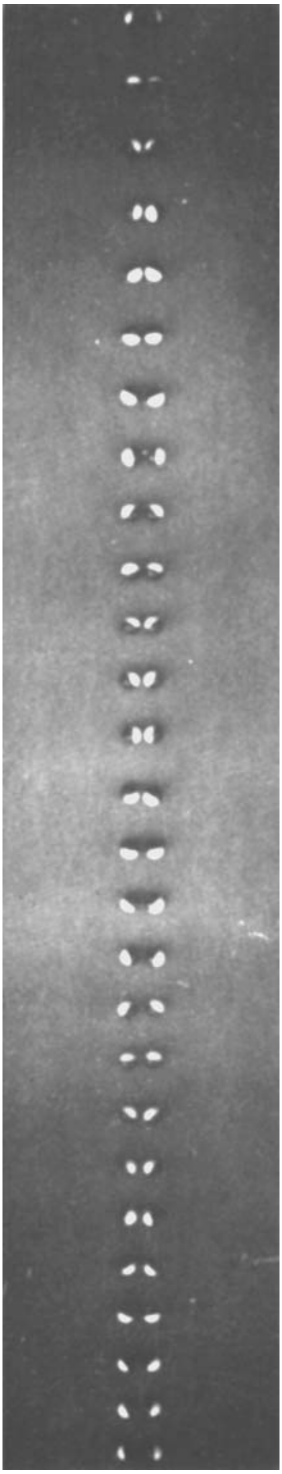

FIGURE 2. The sedimentation of a pair of half-painted, equal-size spheres, showing that they rotate and separate as they fall. Perspex spheres, $d = \frac{3}{8}$ in., $Re \simeq 0{\cdot}1$, photographed at intervals of 2 sec.

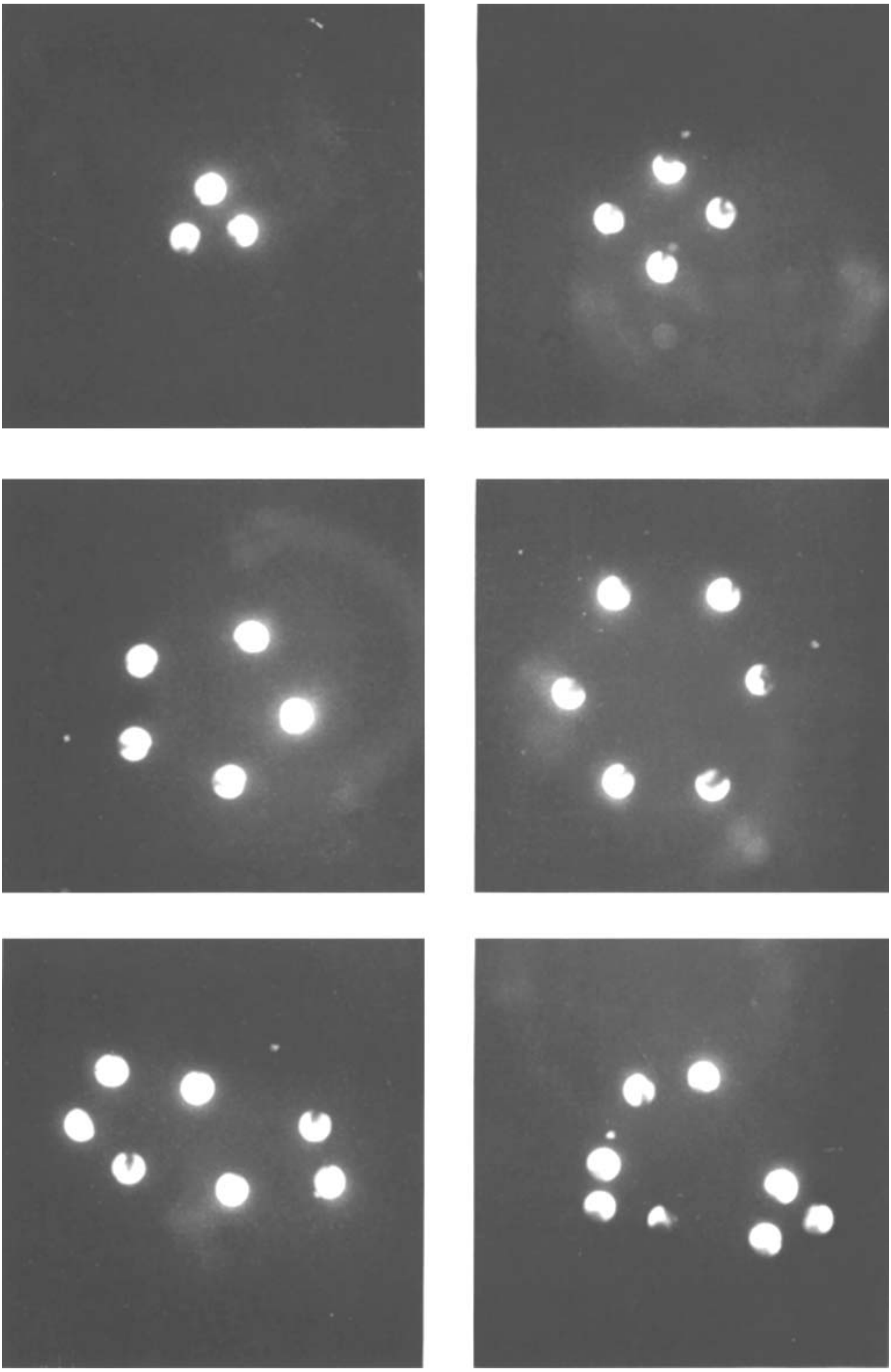

FIGURE 5. Clusters of 3 to 6 equal spheres, with $0{\cdot}06 < Re < 7$, arrange themselves in the same horizontal plane at the vertices of regular polygons. Clusters of 7 or 8 spheres fail to form regular configurations.